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ASYMPTOTIC AND VARIATIONAL METHODS IN NON-LINEAR PROBLEMS OF THE INTERACTION OF SURFACE WAVES WITH ACOUSTIC FIELDS†

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Potential flows in a system consisting of compressible barotropic ideal fluids – a liquid and a gas with an interface and an acoustic high-frequency vibrator, placed in the gas, are considered. The system of two media completely occupies a bounded absolutely rigid vessel. The two-scale expansion method is applied to the problem in a differential and variational formulation in the Hamilton–Ostrogradskii form. This enables both averaged equations of motion and the principle of the minimum quasi-potential energy to be derived for averaged surface reliefs (capillary-acoustic forms of equilibrium). In the equations obtained and in the functional, terms appear corresponding to forces of vibration origin. The problem of the quasi-equilibrium of the bifurcation of quasi-equilibrium forms is discussed in the case when the plane interface is simultaneously a capillary and a capillary-acoustic equilibrium form. Spectral theorems are derived for the problem of normal oscillations about quasi-equilibrium, and spectral and variational criteria of stability are formulated. © 2001 Elsevier Science Ltd. All rights reserved.

The problem of the interaction of the interface between two bounded fluid media with acoustic fields is related to the description of technological processes where parasitic or deliberate high-frequency vibrations are present. These vibrations can change the character of the surface wave phenomena and lead to stabilization (destabilization) of the mobile boundary. The vibrational actions of mass forces (the vibration of a vessel) were considered in [1–3]. A similar class of problems arises when describing the behaviour of a bounded volume of liquid with a free boundary in standing acoustic fields. A number of experimental and theoretical investigations of a levitating drop [4–6] have revealed a number of new surface physical phenomena, described numerically and also using phenomenological approaches. Another way involves using asymptotic and variational methods, developed for finite-dimensional and infinitely dimensional conservative mechanical problems [3, 7].

In this paper we develop a version of a combined asymptotic and variational theory for the problem of the slashing of a capillary liquid in a vessel due to the action of acoustic fields produced in a gas above the liquid [8]. The idea of this approach was proposed earlier in [9, 10]‡ and successfully developed in [2, 3] to describe surface reliefs in a vibrating vessel. The theory leads to a new class of non-linear stationary boundary-value problems with a free boundary, which are a generalization of the capillary problem. The stability of these equilibrium surfaces may be related to the fact that the spectrum of the problem of relative normal oscillations is positive or with the minimum stationary functional of quasi-potential energy [7].

1. FORMULATION OF THE PROBLEM

Suppose an absolutely rigid vessel is occupied by a gas (of volume $Q_1(t)$) and a liquid (of volume $Q_2(t)$). The mobile simply connected interface $\Sigma(t)$ is under the action of acoustic fields, which are produced by a high-frequency vibrator, situated on the vessel wall and in contact with the gas. The pulsating acoustic motions of the fluid dynamic system can be described using the potential theory of ideal media.§ The corresponding non-linear evolution boundary-value problem relates the velocity potential φ_i, the pressure function p_i and the density ρ_i in the gas ($i = 1$) and in the liquid ($i = 2$) and

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‡See also TIMOKHA, A. N., Direct methods of solving static and eigenvalue problems of the theory of the interaction of surface waves with acoustic fields. Preprint No. 93.15. Inst. Matematiki, Kiev, 1993.

§For a more detailed discussion of the problem of choosing the hydromechanical model see: LUKOVSKII, I. A. and TIMOKHA, A. N., Non-linear dynamics of the interface between a liquid and a gas when the gas contains a high-frequency acoustic field. Steady motions. Preprint No. 88.9. Inst. Matematiki, Kiev, 1988.

can be written in dimensionless form (the linear dimension of the cavity l and the period of the pulsations $2\pi/\nu$ are chosen as the characteristic dimensions and time) as follows:

$$\rho_i \nabla\left(\frac{\partial \varphi_i}{\partial t} + \frac{1}{2}(\nabla\varphi_i)^2 + \nu_*^{-2}\,\mathrm{B}\,x\right) = -\nabla p_i;\ \rho_i = \left(\frac{p_i}{p_{0i}}\right)^{1/\gamma_i} \text{ in } Q_i(t) \tag{1.1}$$

$$\frac{\partial \rho_i}{\partial t} + \operatorname{div}(\rho_i \nabla \varphi_i) = 0 \text{ in } Q_i(t) \tag{1.2}$$

$$\frac{\partial \varphi_i}{\partial n} = 0 \text{ on } S_i;\ \frac{\partial \varphi_i}{\partial n} = -\frac{\xi_t}{|\nabla \xi|} \text{ on } \Sigma(t) \tag{1.3}$$

$$-p_2 + \nu_*^{-2}(K_1 + K_2) = -p_1 \frac{\rho_{01}}{\rho_{02}} \text{ on } \Sigma(t) \tag{1.4}$$

$$-\frac{(\nabla W, \nabla \xi)}{|\nabla W||\nabla \xi|} = \cos\alpha \text{ on } \partial\Sigma(t) \tag{1.5}$$

$$\rho_1 \frac{\partial \varphi_1}{\partial n} = \varepsilon V(x,y,z)\frac{\mu_0}{k}\sin t \text{ on } S_0 \tag{1.6}$$

The cavity of the vessel $Q = Q_1 \cup Q_2$ is specified by the inequality $W(x, y, z) < 0$, $\xi(x, y, z, t) = 0$ is the equation of the interface, $S_i = \partial Q \cap \partial Q_i$ are the vessel walls, in contact with the gas and the liquid respectively, K_i are the principal curvatures of the surface $\Sigma(t)$, $\nu_*^2 = \nu^2\rho_{02}l/\sigma$ is the square of the dimensionless frequency, $\mathrm{B} = gl^2\rho_{02}l/\sigma$ is the Bond number, $k = \nu l/c$ is the wave number of the acoustic field in the gas, c is the velocity of sound in the gas, σ is the surface tension coefficient, $\mathbf{g}$ is the acceleration due to gravity, γ_i are the constants of the Tait equation of state, ν is the vibration frequency, $V_0(x, y, z)\sin(\nu t)$ is the specified distribution of the normal velocities on the acoustic vibrator $S_0 \subset S_1$, ρ_{0i} are the mean densities of the gas and the liquid, and α is the wetting angle. The normal $\mathbf{n}$ to the free surface is the outward normal with respect to the volume of the liquid $Q_2(t)$. In addition $V = V_0/\sup|V_0|$, $\mu_0 = O(1)$ is a dimensionless coefficient of proportionality between the dimensionless amplitude $\varepsilon = \sup|V_0|/(c\mu_0) \ll 1$ and the Mach number of the acoustic vibrations.

Note that the media are barotropic and the first pair of equations admits of a Lagrange–Cauchy type integral. This enables us to get rid of the Tait equation of state, but it does not simplify the technique of two-scale asymptotic expansions used below.

2. ASYMPTOTIC REDUCTION OF THE PROBLEM

The ratio of the density of the gas to the density of the liquid $\rho_{01}/\rho_{02} \ll 1$ and $\nu_*^{-2} \ll 1$ will be assumed to be small quantities. The latter relation is because the frequencies of acoustic vibrations considerably exceed the fundamental mode of the natural oscillations of a capillary liquid in the vessel.

We will assume that the following asymptotic relations between the small parameters are satisfied

$$\rho_{01}/\rho_{02} = \delta = \mu_1\varepsilon,\ |\mu_1| = O(1);\ \nu_*^{-2} = \mu\mu_1\varepsilon^3,\ \mu = O(1) \tag{2.1}$$

Using the method of two-scale expansions [11] we will distinguish the pulsation and slowly oscillating components in problem (1.1)–(1.6). The first is related to the acoustic vibrations of continuous media and has a characteristic period $O(2\pi)$. The slow component is due to the mobility of the interface and depends on the reciprocal potential fields. In the case investigated, the characteristic time of the slow oscillations τ is proportional to the square root of the absolute quantity standing at the potential terms: $\pi = \varepsilon^{3/2}t = \varepsilon_1^3 t$.

Representing the solution of problem (1.1)–(1.6) and (2.1) in the form

$$\varphi_i = \sum_{k=0}^{\infty} \varepsilon_1^k \varphi_i^{(k)}(x,y,z,t,\tau),\ p_i = \sum_{k=0}^{\infty} \varepsilon_1^k p_i^{(k)}(x,y,z,t,\tau)$$

$$\rho_i = \sum_{k=0}^{\infty} \varepsilon_1^{(k)} \rho_i^{(k)}(x,y,z,t,\tau),\ \xi = \sum_{k=0}^{\infty} \varepsilon_1^k \xi_k(x,y,z,t,\tau) \tag{2.2}$$

we derive the averaged problem, which approximately describes the slow oscillations of the system

$$\Delta\varphi = 0 \text{ in } \langle Q_2\rangle(\tau) \tag{2.3}$$

$$\frac{\partial\varphi}{\partial n} = 0 \text{ on } \langle S_2\rangle(\tau) \tag{2.4}$$

$$\frac{\partial\varphi}{\partial n} = -\frac{\zeta_\tau}{|\nabla\zeta|} \text{ on } \langle\Sigma\rangle(\tau) \tag{2.5}$$

$$\varphi_\tau + \frac{1}{2}(\nabla\varphi)^2 + \mu\mu_1(\mathrm{B}x - (K_1 + K_2)) + \frac{\mu_1}{4}(k^2\Phi^2 - (\nabla\Phi)^2) = \text{const on } \langle\Sigma\rangle(\tau) \tag{2.6}$$

$$-\frac{(\nabla W, \nabla\zeta)}{|\nabla W||\nabla\zeta|} = \cos\alpha \text{ on } \partial\langle\Sigma\rangle(\tau); \quad \int\limits_{\langle Q_2\rangle} dQ = \text{const}$$

$$\Delta\Phi + k^2\Phi = 0 \text{ in } \langle Q_1\rangle(\tau)$$

$$\frac{\partial\Phi}{\partial n} = 0 \text{ on } \langle S_1\rangle(\tau) \cup \langle\Sigma\rangle(\tau); \quad \frac{\partial\Phi}{\partial n} = \mu_0\frac{V(x,y,z)}{k} \text{ on } \mathrm{S}_0 \tag{2.7}$$

Here $\langle\cdot\rangle$ denotes averaging over the fast time t.

Boundary-value problem (2.3)–(2.7) relates the principal terms of the asymptotic expansion (2.2)

$$\varphi_2 = \varepsilon_1^3\varphi(x,y,z,\tau) + o(\varepsilon_1^3)$$
$$\varphi_1 = \varepsilon_1^2\Phi(x,y,z,\tau)\sin t + O(\varepsilon_1^3); \quad \xi = \zeta(x,y,z,\tau) + o(\varepsilon_1^3) \tag{2.8}$$

Boundary-value problem (2.3)–(2.6) is identical in form with the problem of waves on the surface of a capillary liquid [12–15]. The additional pseudo-differential terms in the dynamic boundary condition (2.6) have the meaning of the pressure (the acoustic radiation pressure), applied to the free surface. This ressure depends parametrically on the position of the surface $\langle\Sigma\rangle(\tau)$($\Phi$ is the solution of Neyman problem (2.7) with a mobile boundary).

3. THE PROBLEM OF THE CAPILLARY-ACOUSTIC EQUILIBRIUM FORM

If the position of the average surface is independent of τ, i.e.

$$\langle\Sigma\rangle = \Sigma_0 : \zeta_0 = \zeta_0(x,y,z) = 0, \ \langle Q_1\rangle = Q_0, \ \varphi = 0, \ \Phi = \Phi_0(x,y,z)$$

problem (2.3)–(2.7) is reduced to the stationary boundary-value problem

$$-\mu(K_1 + K_2) - \mu\,\mathrm{B}x + \frac{1}{4}(k^2\Phi_0^2 - (\nabla\Phi_0)^2) = \text{const on } \Sigma_0 \tag{3.1}$$

$$-\frac{(\nabla W, \nabla\zeta_0)}{|\nabla W||\nabla\zeta_0|} = \cos \text{ on } \partial\Sigma_0; \quad \int\limits_{\langle Q_2\rangle} dQ = \text{const}$$

where Φ_0 satisfies the problem

$$\Delta\Phi_0 + k^2\Phi_0 = 0 \text{ in } Q_0$$

$$\frac{\partial\Phi_0}{\partial n} = 0 \text{ on } \langle S_1\rangle \cup \Sigma_0; \quad \frac{\partial\Phi_0}{\partial n} = \mu_0\frac{V(x,y,z)}{k} \text{ on } S_0 \tag{3.2}$$

Equation (3.1) expresses the balance between the capillary, gravitational and acoustic radiation forces. By analogy with the capillary form of equilibrium, we have called the surface Σ_0, found from (3.1) and (3.2), the capillary-acoustic equilibrium form. For the asymptotic relations (3.1) the geometry of the capillary-acoustic surface may differ considerably from the geometry of the capillary surface. The nature of the sloshing of the liquid and the stability of the interface between the two phases changes correspondingly.

4. THE THEORY OF LINEAR NORMAL OSCILLATIONS ABOUT A CAPILLARY-ACOUSTIC EQUILIBRIUM FORM. THE STABILITY OF QUASI-EQUILIBRIUM FORMS

We will linearize problem (2.3)–(2.7) with respect to the capillary-acoustic equilibrium form Σ_0: $x = H_0(y, z)$ and seek normal oscillations of the form

$$h = \exp(i\omega\tau)\, H(y, z);\ \varphi = i\omega \exp(i\omega\tau)\psi(x, y, z);$$
$$\Phi = i\omega \exp(i\omega\tau)\Psi(x, y, z)$$

This leads to the following eigenvalue boundary-value problem in H and ψ

$$\Delta\psi = 0 \text{ in } \langle Q_2\rangle;\ \frac{\partial\psi}{\partial n} = 0 \text{ on } \langle S_2\rangle;\ \frac{\partial\psi}{\partial n} = \frac{H}{(1+(\nabla H_0)^2)^{1/2}} \text{ on } \Sigma_0 \tag{4.1}$$

$$-\omega^2\psi + \mu_1\mu A H = 0 \text{ on } \Sigma_0 \tag{4.2}$$

with spectral parameter ω^2. The linear operator $A = A_1 + A_2$ has the form

$$AH = [A_1 H] + [A_2 H] = \left[-\operatorname{div}\left\{\frac{\nabla H}{(1+(\nabla H_0)^2)^{1/2}} - \frac{(\nabla H, \nabla H_0)\nabla H_0}{(1+(\nabla H_0)^2)^{3/2}}\right\}\right] +$$
$$+\left[\frac{1}{2\mu}\left\{k^2\Phi_0\Phi_{0x}H - (\nabla\Phi_0, \nabla\Phi_{0x})H + k^2\Phi_0\Psi - (\nabla\Phi_0, \nabla\Psi)\right\} + \mathrm{B}\, H\right] \tag{4.3}$$

$$\frac{W_y H_y + W_z H_z}{|\nabla_2 W|} = \frac{W_y H_{0y} + W_z H_{0z}}{|\nabla_2 W|}\frac{(\nabla H, \nabla H_0)}{(1+(\nabla H_0)^2)^{1/2}} \text{ on } \partial\Sigma_0;\ \int_{\Sigma_0} H dy dz = 0$$
$$\Delta\Psi + k^2\Psi = 0 \text{ in } Q_0;\ \frac{\partial\Psi}{\partial n} = 0 \text{ on } \langle S_1\rangle \cup S_0 \tag{4.4}$$
$$\frac{\partial\Psi}{\partial n} = \frac{\Phi_{0xx}H - \Phi_{0z}H_z - \Phi_{0y}H_y - [\Phi_{0xy}H_{0y} + \Phi_{0xz}H_{0z}]H}{(1+(\nabla H_0)^2)^{1/2}} \text{ on } \Sigma_0$$

We investigated the eigenvalue properties of the pseudo-differential operator A and we showed that it is self-conjugate and has a real point spectrum with a finite number of negative eigenvalues (see the publication cited in the footnote ‡ on page 463).

The following theorem establishes the main properties of eigenvalue problem (4.1), (4.2) for the operator (4.3), (4.4).

Theorem 1. Suppose H_0, Φ_0 is the solution of the problem of the capillary-acoustic equilibrium form (2.3)–(2.7) where $H_0 \in C^2(P\Sigma_0)$, $\Phi_0 \in C^2(Q_0 \cup \Sigma_0)$ ($P\Sigma_0$ is the projection of Σ_0 on Oyz). Then 1) eigenvalue problem (4.1)–(4.4) has a real point spectrum consisting of eigenvalues and $\{H_n\}$ is the basis in factor-space $L_2(P\Sigma_0)$/const; 2) the set of negative eigenvalues ω_n^2 is finite.

Proof. We introduce the auxiliary operator T: $H \to \psi|_{\Sigma_0}$, defined by the solution of Neyman problem (4.1). The operator T is precompact and reversible on the everywhere dense set in $L_2(P\Sigma_0)$/const. Boundary condition (4.2) leads to the eigenvalue equation

$$C_0(\omega^2)H = (\mu\mu_1 A - \omega^2 T)H = 0 \tag{4.5}$$

with eigenvalue parameter ω^2. The set of its solutions is identical with the spectrum of the initial problem (4.1)–(4.4).

Consider the operator A_1 defined by (4.3). It arises when analysing the normal oscillations of a capillary liquid and is unbounded, self-conjugate and positive in $L_2(P\Sigma_0)$/const [14]. We will introduce the auxiliary operators C_1 and C_2 as follows:

$$C_1(\omega^2) = \omega^2 A_1^{-1} T - \mu\mu_1(E + A_1^{-1} A_2) = C_2(\omega^2) - \mu\mu_1 E$$

where C_1 is obtained by the action of A_1^{-1} from the left on operator C_0 of Eq. (4.5). The operator $C_2(\omega^2)$ is precompact in $L_2(P\Sigma_0)$/const. Hence, if ω^2 is a solution of Eq. (4.5), then $\mu\mu_1$ is an eigenvalue of operator C_2 and, consequently, ω^2 is an eigenvalue of eigenvalue problem (4.1)–(4.4). Since T and A are self-conjugate operators, these eigenvalues are real.

The regular set of eigenvalue problem (4.1)–(4.4) is not empty and contains at least complex numbers with a non-zero imaginary component. For the regular point ω_0^2, Eq. (4.5) is equivalent to an eigenvalue equation of the form

$$(C + (\omega^2 - \omega_0^2)^{-1} E)H = 0$$

where $C(\omega_0^2) = C_1(\omega_0^2)^{-1} A_1^{-1} T$ is a completely continuous operator in $L_2(P\Sigma_0)$. Since the operator C is compact, its point spectrum consists of eigenvalues and Assertion 1 of the theorem holds.

All the eigenvalues of the operator $A_1^{-1} T$ are positive like the eigenvalues of the problem of the oscillations of a capillary liquid, i.e. for all permissible H

$$(A_1^{-1} TH, H) > 0$$

Then

$$\omega_n^2 = \mu\mu_1((H_n, H_n) + (A_1^{-1} A_2 H_n, H_n))/(A_1^{-1} TH_n, H_n)$$

where $(H_n, H_n) = 1$, $(A_1^{-1}\, TH_n, H_n) > 0$. Since the operator $A_1^{-1} A_2$ is compact and $\{H_n\}$ is a basis in $L_2(P\Sigma_0)$, then $(A_1^{-1} A_2 H_n, H_n) \to 0$, $n \to \infty$. Consequently Assertion 2 is true.

Corollary 1. Loss of stability of a capillary-acoustic equilibrium form can only occur by a finite number of linearly independent modes (perturbations).

Corollary 2. A capillary-acoustic equilibrium form is stable if and only if all the eigenvalues of the operator A are positive.

The second corollary is identical in form with the eigenvalue principle of stability, used previously to analyse the stability of a capillary form of equilibrium [14] (the stability was investigated using the eigenvalue properties of an operator of typle A_1).

5. A PLANE CAPILLARY-ACOUSTIC EQUILIBRIUM FORM

A plane capillary surface in a straight cylindrical vessel is obtained when the wetting angle is right. This surface remains plane if the acoustic vibrator at the upper end produces a plane acoustic wave

$$V_0(x, y, z) = V_0 = \text{const}$$

$$\left(\varepsilon = -\frac{V_0}{c \sin(kh_1)},\ \mu_0 = -\sin(kh_1), V(y, z) = 1\right)$$

The problem of the capillary-acoustic equilibrium form then has the "trivial" solution

$$H_0(y, z) \equiv 0;\ \Phi_0(x, y, z) = k^{-2} \cos(kx) \tag{5.1}$$

This model case is convenient for comparing the properties of the capillary form and the capillary-acoustic equilibrium form. It is well known (see [14, 16]), that a plane capillary surface corresponds to the unique solution of the capillary problem in a vessel in the form of a circular cylinder, if $B > \varkappa_{11}^2$, where $\varkappa_{11}$ is the minimum root of the equation $J_1'(\varkappa_{11}) = 0$ ($J_p(\cdot)$ are Bessel functions). We will expand the solution of non-linear boundary-value problem (3.1), (3.2) in a Fourier series in the complete system of functions

$$h_{pq}(r, \theta) = J_p(\varkappa_{pq} r) \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (p\theta)$$

(in a polar system of coordinates). We obtain

$$H_0(r,\theta) = \sum_{pq \neq 00} \eta_{pq} h_{pq}(r,\theta) \tag{5.2}$$

$$\Phi_0(x,y,z) = k^{-2}\cos(kx) + \sum_{pq \neq 00} \chi_{pq} b_{pq}(x) h_{pq}(r,\theta) + \chi_{00}\cos(k(x-h_1)) \tag{5.3}$$

$$b_{pq}(x) = \begin{cases} -\dfrac{\mathrm{ch}(\phi(x-h_1))}{\mathrm{ch}(\phi h_1)\phi\,\mathrm{th}(\phi h_1)}, & \varkappa_{pq} > k, \\ -\dfrac{\cos(\phi(x-h_1))}{\cos(\phi h_1)\phi\,\mathrm{tg}(\phi h_1)}, & \varkappa_{pq} < k, \end{cases} \qquad \phi = \sqrt{|\varkappa_{pq}^2 - k^2|}$$

Here η_{pq}, χ_{pq} are unknown coefficients, where two unknown coefficients correspond to each subscript pq in the case of a non-axisymmetrical function $h_{pq}(r, \theta)$ and one subscript in the case of a symmetrical function, i.e.

$$\eta_{pq} h_{pq}(r,\theta) = \begin{cases} \eta'_{pq} J_p(\varkappa_{pq} r)\sin p\theta + \eta''_{pq} J_p(\varkappa_{pq} r)\cos p\theta, & p \neq 0 \\ \eta_{0q} J_0(\varkappa_{0q} r), & p = 0 \end{cases} \tag{5.4}$$

Substituting expansions (5.2) and (5.3) into Eqs (3.1) and (3.2) and using the Fredholm alternative, we obtain an infinite system of non-linear equations in $\eta = \{\eta_{pq}\}$. Up to terms $O(\|\eta\|)$ we have

$$G_{\alpha\beta} = C_{\alpha\beta}\eta_{\alpha\beta} + o(\|\eta\|) = 0 \tag{5.5}$$

$$C_{pq} = \mu(\mathrm{B} + \varkappa_{pq}) + \tfrac{1}{2} b_{pq}(0), \quad p = 0,1,\ldots; \quad q = 1,2,\ldots \tag{5.6}$$

System (5.5) has a "trivial" solution $\eta = 0$ which corresponds to a plane form of equilibrium. In addition, C_{pq} are eigenvalues of the operator A. This means that when $C_{pq} > 0$ the "trivial" solution defines a stable capillary-acoustic equilibrium form. If a subscript pq exists such that $C_{pq}(k) = 0$, the "trivial" solution may be a non-unique solution of the problem of the capillary-acoustic equilibrium form. For eigenvalues with $p \neq 0$, then immediately two equations in (5) have no linear component in η. According to relations (5.4) the eigenvalues $C_{0q}(q = 1, 2, \ldots)$ have unit multiplicity, and Krasnosel'skii's theorem [17, p.135] gives the sufficient condition for bifurcation of the "trivial" solution.

6. THE VARIATIONAL FORMULATION OF THE PROBLEM AND A MINIMUM PRINCIPLE FOR CAPILLARY-ACOUSTIC EQUILIBRIUM FORMS

The example given above shows that the determination of stable capillary-acoustic equilibrium forms, starting from differential formulation (3.1), (3.2), can be effective when the forms investigated are identical with capillary forms. If the capillary-acoustic equilibrium form has a geometry differing from capillary, a more effective method of determining stable forms of equilibrium could be the boundary variation method [14, 15]. However, unlike the problem of the capillary, problem (3.1), (3.2) has no variational analogue (the potential energy functional). It has been shown [3, 10], that a variational problem equivalent to (1.1)–(1.6) can be constructed by four independent methods, which can be conventionally related to the Hamilton–Ostrogradskii, Bateman, Berdichevskii and Luke functionals [18–21]. The use of a variational problem in the Hamilton–Ostrogradskii form, averaged over fast vibrations, enables a variational principle of the stability of capillary-acoustic equilibrium forms to be constructed, which is identical in form with the minimum potential energy principle. The possibility of such an averaged variational formulation has been proved for Hamilton system [7]. Such a functional has been constructed for the case of a vibrating vessel [3]. We will use Theorem 1 from [3] (a Hamilton–Ostrogradskii variational problem), which asserts that the set of continuous solutions of problem (1.1)–(1.6) is identical with the set of stationary points of the functional.

$$G(\xi,\varphi_i,\rho_i)=\int_{t_1}^{t_2}\left\{\int_{Q_2}\rho_2\left(\frac{(\nabla\varphi_2)^2}{2}-U_2(\rho_2)-\mu\mu_1\varepsilon^3\,\mathrm{B}x\right)dQ-\mu\mu_1\varepsilon^3(|\Sigma|-\cos\alpha\,|S_2|)+\right.$$

$$\left.+\varepsilon\int_{Q_1}\rho_1\left(\frac{(\nabla\varphi_1)^2}{2}-U_1(\rho_1)-\mu\mu_1\varepsilon^3\,\mathrm{B}x\right)dQ\right\}dt \tag{6.1}$$

with constraints (1.1)–(1.3) and (1.6) and the conditions for continuous isochronous variations

$$\delta\xi\,|_{t_1,t_2}=0,\quad \delta\rho_i\,|_{t_1,t_2}=0 \tag{6.2}$$

where $p_i=\rho_i^2 dU_i/d\rho_i$.

Using the two-scale expansion technique [3], it can be shown that

$$\langle G(\xi,\varphi_i,\rho_i)\rangle=\mathrm{const}+\varepsilon_1^3\,\mathcal{G}(\zeta,\psi)+O(\varepsilon_1^4)$$

where

$$\mathcal{G}(\zeta,\psi)=\int_{\tau_1}^{\tau_2}\left\{\int_{\langle Q_2\rangle}\left(\frac{(\nabla\varphi)^2}{2}-\mu\mu_1\,\mathrm{B}x\right)dQ-\mu\mu_1(|\langle\Sigma\rangle|-\cos\alpha\,|\langle S_2\rangle|)+\right.$$

$$\left.+\frac{\mu_1}{4}\int_{\langle Q_1\rangle}(k^2\Phi^2-(\nabla\Phi)^2)dQ-\frac{\mu_0\mu_1}{2k}\int_{S_0}\Phi V(x,y,z)ds\right\}d\tau \tag{6.3}$$

The following theorem expresses the minimum principle for capillary-acoustic equilibrium forms.

Theorem 2. The problem of determining stable continuous solutions of the problem of a capillary-acoustic equilibrium form Σ_0: $\zeta_0=0$ is equivalent to the problem of finding strict minima of the functional

$$\Pi(\zeta_0)=\mu\left(|\Sigma_0|+\cos\alpha\,|\langle S_1\rangle|+\int_{\langle Q_2\rangle}\mathrm{B}x\,dQ\right)+$$

$$+\left(\frac{1}{4}\int_{Q_0}(k^2\Phi_0^2-(\nabla\Phi_0)^2)dQ+\frac{\mu_0}{2k}\int_{S_0}V(x,y,z)\Phi_0 ds\right)=-\mathcal{G}(\zeta_0(x,y,z),\ \Phi_0(x,y,z))$$

where Φ_0 is the solution of boundary-value problem (3.2), and the condition of conservation of volume

$$\int_{\langle Q_2\rangle}dQ=\mathrm{const}$$

is satisfied.

The proof is carried out by calculating the second variation of the function $\Pi(\zeta_0)$ when $\zeta_0=x-H_0$. The second variation with respect to Σ_0 was calculated previously in [14, Chapter 1] (the first variation with respect to Φ_0, as can easily be shown, is equal to zero in the case of constraint (3.2), while the first variation with respect to ζ_0 leads to Eqs (3.1)), which connects Φ_0 and ζ_0. Here

$$\delta^2\Pi=\mu^{-1}\int_{P\Sigma_0}(A\delta H,\delta H)dydz$$

where A is operator (4.3), (4.4). The condition $\delta^2\Pi>0$ is equivalent to the eigenvalue stability principle.

Note that the use of the variational approach enables the idea of a generalized solution of the problem to be introduced. A rigorous mathematical solution of the problem of constructing a theory of generalized solutions of problem (3.1), (3.2) is the subject of additional mathematical investigations.

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